The 1-D Wave Equation

18.303 Linear Partial Differential Equations

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1 1-D Wave Equation : Physical derivation

Reference: Haberman §4.1-4.4

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We consider a string of length l with ends fixed, and rest state coinciding with x-axis. The string is plucked into oscillation. Let $u(x,t)$ be the position of the string at time t.

Assumptions:

1. Small oscillations, i.e. the displacement $u(x,t)$ is small compared to the length l.
   (a) Points move vertically. In general, we don't know that points on the string move vertically. By assuming the oscillations are small, we assume the points move vertically.
   (b) Slope of tangent to the string is small everywhere, i.e. $|u_x(x,t)| \ll 1$, so stretching of the string is negligible
   (c) arc length $\alpha(t) = \int_0^l \sqrt{1+u_x^2}dx \simeq l$.
2. String is perfectly flexible (it bends). This implies the tension is in the tangent direction and the horizontal tension is constant, or else there would be a preferred direction of motion for the string.

Consider an element of the string between x and $x + \Delta x$. Let $T(x,t)$ be tension and $\theta(x,t)$ be the angle wrt the horizontal x-axis. Note that

$$\tan\theta(x,t) = \text{slope of tangent at } (x,t) \text{ in } ux\text{-plane} = \frac{\partial u}{\partial x}(x,t). \tag{1}$$

Newton's Second Law ($F = ma$) states that

$$F = (\rho \Delta x) \frac{\partial^2 u}{\partial t^2} \tag{2}$$

where ρ is the linear density of the string (ML^{-1}) and Δx is the length of the segment. The force comes from the tension in the spring only - we ignore any external forces such as gravity. The horizontal tension is constant, and hence it is the vertical tension that moves the string vertically (obvious).

Balancing the forces in the horizontal direction gives

$$T(x + \Delta x, t) \cos\theta(x + \Delta x, t) = T(x, t) \cos\theta(x, t) = \tau = const \tag{3}$$

where τ is the constant horizontal tension. Balancing the forces in the vertical direction yields

$$\begin{aligned} F &= T(x + \Delta x, t) \sin\theta(x + \Delta x, t) - T(x, t) \sin\theta(x, t) \\ &= T(x + \Delta x, t) \cos\theta(x + \Delta x, t) \tan\theta(x + \Delta x, t) - T(x, t) \cos\theta(x, t) \tan\theta(x, t) \end{aligned}$$

Substituting (3) and (1) yields

$$\begin{aligned} F &= \tau \left(\tan\theta(x + \Delta x, t) - \tan\theta(x, t)\right) \\ &= \tau \left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right). \end{aligned} \tag{4}$$

Substituting F from (2) into Eq. (4) and dividing by Δx gives

$$\rho \frac{\partial^2 u}{\partial t^2}(\xi, t) = \tau \frac{\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)}{\Delta x}$$

for $\xi \in [x, x + \Delta x]$. Letting $\Delta x \to 0$ gives the 1-D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{\tau}{\rho} > 0. \tag{5}$$

Note that c has units $[c] = \left[\frac{\text{Force}}{\text{Density}}\right]^{1/2} = LT^{-1}$ of speed.

1.1 Boundary conditions

In order to guarantee that Eq. (5) has a unique solution, we need initial and boundary conditions on the displacement $u(x, t)$. There are now 2 initial conditions and 2 boundary conditions.

E.g. The string is fixed at both ends, $u(0, t) = u(l, t) = 0$, $t > 0$ (homogeneous type I BCs)

E.g. The string is connected to frictionless cylinders of mass m that move vertically on tracks at $x = 0, l$. Performing a force balance at either $x = 0$ or $x = 1$ gives

$$T \sin \theta = mg \tag{6}$$

In other words, the vertical tension in the string balances the mass of the cylinder. But $\tau = T \cos \theta = const$ and $\tan \theta = u_x$, so that (6) becomes

$$\tau u_x = T \cos \theta \tan \theta = mg$$

Rearranging yields

$$u_x = \frac{mg}{\tau}, \qquad x = 0, 1$$

These are Type II BCs. If the string is really tight and the cylinders are very light, then $mg/\tau \ll 1$ and we approximate $u_x \approx 0$ at $x = 0, 1$, and the BCs become Type II homogeneous BCs.

1.2 Initial conditions

E.g. Prescribe initial position of string and initial velocity: $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$, $0 < x < l$.

To see why we need 2 initial conditions, note that the Taylor series of $u(x, t)$ about $t = 0$ is

$$u(x, t) = u(x, 0) + u_t(x, 0)\, t + u_{tt}(x, 0) \frac{t^2}{2} + \frac{\partial^3 u}{\partial t^3}(x, 0) \frac{t^3}{3!} + \cdots$$

From the initial conditions, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ and the PDE gives

$$\begin{aligned} u_{tt}(x, 0) &= c^2 u_{xx}(x, 0) = c^2 f''(x), \\ \frac{\partial^3 u}{\partial t^3}(x, 0) &= c^2 u_{txx}(x, 0) = c^2 g''(x). \end{aligned}$$

Higher order terms can be found similarly. Therefore, the two initial conditions for $u(x, 0)$ and $u_t(x, 0)$ are sufficient to determine $u(x, t)$ near $t = 0$.

To summarize, the dimensional basic 1-D Wave Problem is

$$\text{PDE} : \quad u_{tt} = c^2 u_{xx}, \qquad 0 < x < l \tag{7}$$

$$\text{BC} : \quad u(0, t) = 0 = u(l, t), \qquad t > 0, \tag{8}$$

$$\text{IC} : \quad u(x, 0) = f(x), \qquad u_t(x, 0) = g(x), \qquad 0 < x < l \tag{9}$$

1.3 Non-dimensionalization

We now scale the basic 1-D Wave Problem. The length of the string motivates scaling x via

$$\hat{x} = \frac{x}{l}.$$

We introduce the scaled time as $\hat{t} = c_1 t$, where c_1 is to be determined. We write $\hat{u}\left(\hat{x}, \hat{t}\right) = u\left(x, t\right)$. Using the chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial \hat{u}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} = \frac{1}{l}\frac{\partial \hat{u}}{\partial \hat{x}}, \qquad \frac{\partial u}{\partial t} = \frac{\partial \hat{u}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} = c_1\frac{\partial \hat{u}}{\partial \hat{t}}$$

and similarly for higher derivatives. Substituting the dimensionless variables into the 1-D Wave Equation (7) gives

$$c_1^2 \hat{u}_{\hat{t}\hat{t}} = \frac{c^2}{l^2}\hat{u}_{\hat{x}\hat{x}}$$

To best clean up the PDE, we choose $c_1 = c/l$, so that the PDE becomes

$$\hat{u}_{\hat{t}\hat{t}} = \hat{u}_{\hat{x}\hat{x}}, \qquad 0 < \hat{x} < 1, \qquad \hat{t} > 0. \tag{10}$$

The BCs (8) become

$$\hat{u}\left(0, \hat{t}\right) = 0 = \hat{u}\left(1, \hat{t}\right), \qquad \hat{t} > 0. \tag{11}$$

Writing $\hat{f}\left(\hat{x}\right) = f\left(x\right)$ and $\hat{g}\left(\hat{x}\right) = \left(l/c\right) g\left(x\right)$, the ICs (9) become

$$\hat{u}\left(\hat{x}, 0\right) = \hat{f}\left(\hat{x}\right), \qquad \hat{u}_{\hat{t}}\left(\hat{x}, 0\right) = \hat{g}\left(\hat{x}\right), \qquad 0 < \hat{x} < 1. \tag{12}$$

1.3.1 Dimensionless 1-D Wave Problem

Dropping hats, the dimensionless 1-D Wave Problem is, from (10) – (12),

$$\text{PDE}: \qquad u_{tt} = u_{xx}, \qquad 0 < x < 1 \tag{13}$$

$$\text{BC}: \qquad u\left(0, t\right) = 0 = u\left(1, t\right), \qquad t > 0, \tag{14}$$

$$\text{IC}: \qquad u\left(x, 0\right) = f\left(x\right), \qquad u_t\left(x, 0\right) = g\left(x\right), \qquad 0 < x < 1 \tag{15}$$

2 Separation of variables

[Oct 4, 2004]

Substituting $u\left(x, t\right) = X\left(x\right) T\left(t\right)$ into the PDE (13) and dividing by $X\left(x\right) T\left(t\right)$ gives

$$\frac{T''\left(t\right)}{T\left(t\right)} = \frac{X''\left(x\right)}{X\left(x\right)} = -\lambda \tag{16}$$

where λ is a constant. The negative sign is for convention. The BCs (14) become

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \\ u(1,t) &= X(1)T(t) = 0 \end{aligned}$$

which implies

$$X(0) = X(1) = 0 \tag{17}$$

Thus, the problem for $X(x)$ is the same Sturm-Liouville Boundary Value Problem as for the Heat Equation,

$$X''(x) + \lambda X(x) = 0; \qquad X(0) = X(1) = 0. \tag{18}$$

Recall that the eigenvalues and eigenfunctions of (18) are

$$\lambda_n = n\pi, \qquad X_n(x) = b_n \sin(n\pi x), \qquad n = 1, 2, 3, ...$$

The function $T(t)$ satisfies

$$T'' + \lambda T = 0$$

and hence each eigenvalue λ_n corresponds to a solution $T_n(t)$

$$T_n(t) = \alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t).$$

Thus, a solution to the PDE and BCs is

$$u_n(x,t) = (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

where we have absorbed the constant b_n into α_n, β_n.

In general, the individual $u_n(x,t)$'s will not satisfy the ICs. Thus we sum infinitely many of them, using the principle of superposition,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

where

$$\begin{aligned} \alpha_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \\ \beta_n &= \frac{2}{n\pi}\int_0^1 g(x)\sin(n\pi x)\,dx \end{aligned}$$

Note: The convergence of this series is hard to show, because we don't have decaying exponentials $e^{-n^2\pi^2 t}$ in the sum terms (more later).

3 Interpretation - Normal modes of vibration

The terms
$$u_n(x,t) = (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t))\sin(n\pi x)$$
for $n = 1, 2, 3, ...$ are called the normal modes of vibration. The solution $u(x,t)$ is a superposition of the normal modes $u_n(x,t)$. In physical variables, the normal modes are
$$u_n'(x',t') = \left(\alpha_n \cos\left(\frac{n\pi c}{l}t'\right) + \beta_n \sin\left(\frac{n\pi c}{l}t'\right)\right)\sin\left(\frac{n\pi x'}{l}\right).$$

3.1 Frequency and Period

A function $f(t)$ is periodic if for some real number T,
$$f(t+T) = f(t)$$
for all t. If t measures time (either physical or dimensionless), we define the period of a function as the smallest number T such that $f(t+T) = f(t)$. We say that $f(t)$ has period T or, equivalently, that $f(t)$ is T-periodic. The period T has the same dimensions as t. If t is dimensionless then so is T; if t has the dimensions of time, so does T. Note that each normal mode $u_n(x,t)$ has period $2/n$, and in physical variables, $u_n'(x',t')$ has period $2l/(nc)$.

The analog to period for spatial coordinates is the wavelength. The wavelength of a function $g(x)$ (x is a scaled or physical spatial coordinate) is defined as the smallest L such that $g(x+L) = g(x)$. We say $g(x)$ is L-periodic. Again, the wavelength L has the same dimensions as x. If x is dimensionless then so is L; if x has the dimensions of length, so does L.

The frequency f is defined as $f = 1/T = 1/\text{period}$. In physical variables, each normal mode $u_n'(x',t')$ has frequency
$$f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{\tau}{\rho}}$$
with dimensions 1/time (e.g. Hz = cycle/sec = sec^{-1}).

The angular frequency is defined as $\omega = 2\pi f$ where f is the frequency. Each mode has angular frequency $\omega_n = 2\pi f_n = n\pi c/l$. In terms of the frequency, we can write the normal mode as
$$\begin{aligned} u_n'(x',t') &= \left(\alpha_n \cos\left(\frac{\pi n c}{l}t'\right) + \beta_n \sin\left(\frac{n\pi c}{l}t'\right)\right)\sin\left(\frac{n\pi x'}{l}\right) \\ &= (\alpha_n \cos(2\pi f_n t') + \beta_n \sin(2\pi f_n t'))\sin\left(\frac{n\pi x'}{l}\right) \\ &= (\alpha_n \cos(\omega_n t') + \beta_n \sin(\omega_n t'))\sin\left(\frac{n\pi x'}{l}\right) \end{aligned}$$

The first harmonic is the normal mode of lowest frequency, $u_1(x,t)$ or in physical variables, $u_1'(x',t')$.

The fundamental frequency is f_1, i.e. the frequency of the first harmonic. In dimensionless variables, $f_1 = 1/2$. In physical variables, $f_1' = c/(2l) = \left(\sqrt{\tau/\rho}\right)/2l$.

Note that $f_n = nf_1$, in other words, the frequencies of higher harmonics are just integer multiples of the fundamental frequency f_1.

Examples. Check for yourself that $\sin(\pi x)$ is 2-periodic i.e. has period 2, $\sin(x)$ is 2π-periodic, $\sin(n\pi t)$ is $2/n$-periodic, $\sin(n\pi ct/l)$ has period $2l/(nc)$, $\sin(n\pi x/l)$ has period $2l/n$. Note that if a function has period T, then $f(t+mT) = f(t)$ for all $m = 1, 2, 3, ...$ In particular,

$$\sin(n\pi x), \cos(n\pi x), \sin(n\pi t), \cos(n\pi t)$$

are all $2/n$-periodic, and hence

$$\begin{aligned} \sin\left(n\pi\left(x+\frac{2m}{n}\right)\right) &= \sin n\pi x, \qquad \cos\left(n\pi\left(x+\frac{2m}{n}\right)\right) = \cos n\pi x \\ \sin\left(n\pi\left(t+\frac{2m}{n}\right)\right) &= \sin n\pi t, \qquad \cos\left(n\pi\left(t+\frac{2m}{n}\right)\right) = \cos n\pi t \end{aligned}$$

for all $m, n = 1, 2, 3, ..$ Therefore, the dimensionless solution $u(x,t)$ of the wave equation has time period 2 ($u(x,t+2) = u(x,t)$) since

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

and for each normal mode, $u_n(x,t) = u_n(x,t+2)$ (check for yourself). Thus, the period of $u(x,t)$ is the same as that for the first harmonic $u_1(x,t)$. In physical variables, the period of $u'(x',t')$ is $T_1' = 2l/c = 2l/\sqrt{\tau/\rho}$ and the frequency is $f_1' = 1/T_1' = c/(2l)$.

3.2 A note on dimensions

I've tried to denote dimensional quantities with primes or specifically comment whether or not we are working with physical x, t (a length and time) or with dimensionless x, t (dimensions 1). However, in general you should always ask the question, "What are the dimensions?" The quantity c/l has dimensions of 1/time since I have defined c, l to be a speed and a length. The argument of any mathematical function like cos, sin, exp, etc. must be dimensionless. The cosine of 1 m or 2 s does not make sense. Thus, the quantity t in $\cos(n\pi t)$ is dimensionless. Note that radians are a dimensionless measure of angle. Consider the quantity t in $\cos(\omega t)$ or $\cos(2\pi ft)$ or $\cos\left(\frac{\pi ct}{l}\right)$, where you are given that ω and f have dimensions of 1/time, c is a speed, and l is a length. Then t must have dimensions of time, so that the arguments ωt, $2\pi f t$, or $\pi c t/l$ of cosine are dimensionless.

3.3 Amplitude

Note that $u_n(x,t)$ can be written

$$u_n(x,t) = \gamma_n \sin(n\pi x)\sin(n\pi t + \psi_n) \tag{19}$$

where $\gamma_n = \sqrt{\alpha_n^2 + \beta_n^2}$, $\psi_n = \arctan\left(\frac{\alpha_n}{\beta_n}\right)$. At each point x, the n'th mode vibrates sinusoidally according to

$$u_n(x,t) = A_n(x)\sin(n\pi t + \psi_n)$$

where

$$A_n(x) = \gamma_n \left|\sin(n\pi x)\right|.$$

The mode $u_n(x,t)$ vibrates sinusoidally in time between the two limits $\pm A_n(x)$. We call $A(x)$ the time amplitude of the mode $u_n(x,t)$, since this sets the bounds on the oscillations in time.

4 Conservative system and energy

Recall the solution to the Heat Problem with a homogeneous PDE (i.e. no sources, sinks) Homogenous Type I BCs ($u = 0$ at $x = 0, 1$) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

and as $t \to \infty$, $u(x,t) \to 0$. Thus, the rod looses heat energy and with it the memory of the initial state, or initial condition.

In contrast, the solution to the wave equation with Homogeneous Type I BCs (fixed ends, $u = 0$ at $x = 0, 1$),

$$u(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t))\sin(n\pi x)$$

The oscillations do not decay, since we ignored gravity and resistive forces. Once plucked, the string vibrates/oscillates forever, with period $2l/c$ in physical coordinates. We call this type of system conservative: energy is conserved. In addition, the system comes back to its initial condition periodically - i.e. it maintains a memory of its initial state.

We define the energy of the system as

$$\text{Total energy } E'(t') = PE'(t') + KE'(t')$$

where primes denote physical or dimensional variables. The local kinetic energy is

$$KE'(x', t') = \frac{1}{2}mv^2 = \frac{1}{2}(\rho\Delta x)\left(\frac{\partial u}{\partial t}\right)^2$$

and the local potential energy is

$$\begin{aligned} PE'(x', t') &= \text{work done getting to displacement } u \\ &= \text{Force } \times \text{ change in length of string} \end{aligned}$$

Making a displacement Δu of a segment of string from x to $x+\Delta x$ results in a change in the string segment length, initially of length Δx, of

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta u)^2} - \Delta x = \Delta x\left(\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} - 1\right)$$

Recall the binomial expansion $\sqrt{1+a^2} = 1 + \frac{1}{2}a^2 + O(a^4)$. Then

$$\Delta l = \Delta x\left(1 + \frac{1}{2}\left(\frac{\Delta u}{\Delta x}\right)^2 - 1 + O\left(\left(\frac{\Delta u}{\Delta x}\right)^4\right)\right) \approx \frac{\Delta x}{2}\left(\frac{\Delta u}{\Delta x}\right)^2$$

and hence the potential energy is

$$PE'(x', t') = \tau\Delta l \approx \frac{\tau}{2}\left(\frac{\Delta u}{\Delta x}\right)^2 \Delta x \approx \frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2 \Delta x$$

as $\Delta x \to 0$. Thus, the total energy is, in dimensional form,

$$E'(t') = \frac{1}{2}\int_0^l \left(\rho\left(\frac{\partial u}{\partial t'}\right)^2 + \tau\left(\frac{\partial u}{\partial x'}\right)^2\right) dx' = \frac{\rho}{2}\int_0^l \left(u_{t'}^2 + c^2 u_{x'}^2\right) dx' \qquad (20)$$

where we use the shorthand notation $u_{t'}^2 = (\partial u/\partial t')^2$. Substituting $c^2 = \tau/\rho$ and the dimensionless variables into (20) gives the dimensionless total energy

$$E(t) = \frac{E'(t')}{\tau/l} = \frac{1}{2}\int_0^1 \left(u_t^2 + u_x^2\right) dx \qquad (21)$$

The energy of a given normal mode $u_n(x,t)$ is found by substituting the form (19) into Eq. (21) for the energy,

$$\begin{aligned} E_n(t) &= \frac{1}{2}\int_0^1 \left(\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2 \right) dx \\ &= \frac{(n\pi\gamma_n)^2}{2}\int_0^1 \left(\sin^2(n\pi x)\cos^2(n\pi t+\psi_n) + \cos^2(n\pi x)\sin^2(n\pi t+\psi_n)\right) dx \\ &= \frac{(n\pi\gamma_n)^2}{2}\left(\cos^2(n\pi t+\psi_n)\int_0^1 \sin^2(n\pi x)\,dx \right. \\ &\qquad \left. + \sin^2(n\pi t+\psi_n)\int_0^1 \cos^2(n\pi x)\,dx\right) \end{aligned}$$

Note that

$$\sin^2(n\pi x) = \frac{1}{2} - \frac{1}{2}\cos(2n\pi x), \qquad \cos^2(n\pi x) = \frac{1}{2} + \frac{1}{2}\cos(2n\pi x)$$

so that

$$\int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}, \qquad \int_0^1 \cos^2(n\pi x)\,dx = \frac{1}{2}$$

Substituting these integrals into the energy $E_n(t)$ gives

$$E_n(t) = \frac{(n\pi\gamma_n)^2}{4}\left(\cos^2(n\pi t+\psi_n) + \sin^2(n\pi t+\psi_n)\right) = \frac{(n\pi\gamma_n)^2}{4}.$$

Thus the total energy (kinetic + potential across the rod) of each mode is constant!

5 D'Alembert's Solution for the wave equation

Ref: Haberman Chapter 12

5.1 Motivation

Note that each normal mode can be written in alternative form:

$$\begin{aligned} u_n(x,t) &= (\alpha_n\cos(n\pi t) + \beta_n\sin(n\pi t))\sin(n\pi x) \\ &= \frac{1}{2}(\alpha_n\sin(n\pi(x-t)) + \beta_n\cos(n\pi(x-t))) \\ &\quad + \frac{1}{2}(\alpha_n\sin(n\pi(x+t)) + \beta_n\cos(n\pi(x+t))) \end{aligned}$$

where we used the trig identities

$$\begin{aligned} \sin(a+b) + \sin(a-b) &= 2\cos b\sin a \\ \cos(a+b) - \cos(a-b) &= 2\sin a\sin b \end{aligned}$$

Therefore, the mode $u_n(x,t)$ is the sum of a function of $(x-t)$ and a function of $(x+t)$. In physical variables, we can write the solution as the sum of a function of $x'-ct'$ and a function of $x'+ct'$.

5.2 Change of variable

We can define the new coordinates

$$\xi = x - t, \qquad \eta = x + t \tag{22}$$

and let $v(\xi,\eta) = u(x,t)$. The chain rule implies that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \\
\frac{\partial u}{\partial t} &= \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial v}{\partial \xi} + \frac{\partial}{\partial x}\frac{\partial v}{\partial \eta} = \left(\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \xi \partial \eta}\right) + \left(\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) \\
&= \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -\left(-\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \xi \partial \eta}\right) + \left(-\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) \\
&= \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}
\end{aligned}$$

Substituting these derivatives into the PDE $u_{tt} = u_{xx}$ yields

$$\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2} = \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}$$

Simplifying and dividing by 4 gives a new form of the wave equation,

$$\frac{\partial^2 v(\xi,\eta)}{\partial \xi \partial \eta} = 0 \tag{23}$$

5.3 Forward and backward waves

We can write the new form (23) of the wave equation in two ways:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial v(\xi,\eta)}{\partial \eta}\right) = 0, \qquad \frac{\partial}{\partial \eta}\left(\frac{\partial v(\xi,\eta)}{\partial \xi}\right) = 0$$

Integrating the first equation in ξ gives

$$\frac{\partial v(\xi,\eta)}{\partial\eta} = G(\eta) \tag{24}$$

for some function $G(\eta)$ (due to partial integration with respect to ξ). Define the antiderivative of $G(\eta)$ as $Q(\eta)$, i.e. such that $Q'(\eta) = G(\eta)$. Substituting this into (24) and rearranging gives

$$\frac{\partial v(\xi,\eta)}{\partial\eta} = Q'(\eta) \qquad \Longrightarrow \qquad \frac{\partial}{\partial\eta}\left(v(\xi,\eta) - Q(\eta)\right) = 0$$

Integrating the right equation in η yields

$$v(\xi,\eta) - Q(\eta) = P(\xi)$$

where the function $P(\xi)$ is due to the partial integration with respect to η. Therefore, we can write

$$v(\xi,\eta) = P(\xi) + Q(\eta)$$

Substituting for η and ξ from (22) and recalling that $u(x,t) = v(\xi,\eta)$ gives

$$u(x,t) = v(\xi,\eta) = P(x-t) + Q(x+t) \tag{25}$$

5.3.1 Forward wave

The forward wave is the function $P(x-t)$ which represents a wave travelling in the positive x-direction with scaled velocity 1. In physical coordinates, the function depends on $x - ct$ and the speed of the wave is $c = \sqrt{\tau/\rho}$. The shape of the wave is determined by the function $P(x)$ and the motion is governed by the line $x - t = const$. The wave (same shape) moves forward in time along the string.

How do we find the speed from the line $x' - ct' = l_1 = const$? Well, in time $\Delta t'$, we've moved

$$\Delta x' = (l_1 + c(t' + \Delta t')) - (l_1 + ct') = c\Delta t'$$

The speed is distance traveled over elapsed time: $\Delta x'/\Delta t' = c$.

5.3.2 Backward wave

The backward wave is the function $Q(x+t)$ which represents a wave travelling in the negative x-direction with scaled speed 1. The wave shape is determined by $Q(x)$ and the value of the wave is constant along the lines $x + t = const$ (in physical variables, $x' + ct' = const$ and speed is c).

5.4 Characteristics

[Oct 13, 2004]

The solution to the wave equation is the superposition of a forward wave $P(x-t)$ and a backward wave $Q(x+t)$, both with speed c. The lines $x \pm t = const$ are called characteristics.

5.5 Determining the shape functions

The shapes of the forward and backward waves, $P(x)$ and $Q(x)$, are determined from the initial conditions,

$$\begin{aligned} u(x,0) &= f(x) = P(x) + Q(x), && (26) \\ u_t(x,0) &= g(x) = -P'(x) + Q'(x) && (27) \end{aligned}$$

To obtain the second equation, the chain rule was used:

$$\begin{aligned} u_t(x,0) &= \left[\frac{\partial u}{\partial t}(x,t)\right]_{t=0} \\ &= \left[\frac{\partial}{\partial t}\left(P(x-t) + Q(x+t)\right)\right]_{t=0} \\ &= \left[-P'(x-t) + Q'(x+t)\right]_{t=0} \\ &= -P'(x) + Q'(x) \end{aligned}$$

It is important to differentiate in time first and then set $t = 0$ (that is what $u_t(x,0)$ means!). Now, integrating (27) in x from 0 to x gives

$$Q(x) - P(x) - (Q(0) - P(0)) = \int_0^x g(s)\,ds \qquad (28)$$

Eqs. (26) and (28) can be solved for $P(x)$ and $Q(x)$:

$$\begin{aligned} Q(x) &= \frac{1}{2}\left(f(x) + \int_0^x g(s)\,ds + Q(0) - P(0)\right) && (29) \\ P(x) &= \frac{1}{2}\left(f(x) - \int_0^x g(s)\,ds - Q(0) + P(0)\right) && (30) \end{aligned}$$

5.6 D'Alembert's solution to the wave equation

Summarizing our results: from (25), (29) and (30), the wave equation and initial conditions

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

has the solution

$$u(x,t) = P(x-t) + Q(x+t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds \qquad (31)$$

This is called D'Alembert's solution to the wave equation. We have not yet considered the BCs [coming up], so right now we're thinking about an infinite string.

5.6.1 General solution method for infinite string

The general approach to finding $u(x,t)$ given the ICs $f(x)$ and $g(x)$ is:

Step 1. Write down D'Alembert's solution,

$$u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

Step 2. Identify the regions. In general, the function $f(x)$ and $g(x)$ are case functions. You need to determine various regions by plotting the salient characteristics $x \pm t = const$. The regions determine where $x-t$ and $x+t$ are relative to the cases for the functions $f(x)$ and $g(x)$ and tells us what part of the case functions should be used in each region.

Step 3. Determine the solution in each region.

Step 4. For each specific time $t = t_0$, write the x-intervals corresponding to the regions.

5.6.2 Solution method for infinite string, specific ICs

In this section, we consider D'Alembert's solution to the infinite string problem when $f(x)$ and $g(x)$ have the following form,

$$f(x) = \begin{cases} F(x), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}, \qquad g(x) = \begin{cases} G(x), & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \qquad (32)$$

Step 1. D'Alembert's solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

Step 2. Identify the regions. The functions $f(x)$ and $g(x)$ are equal to functions $F(x)$ and $G(x)$, respectively, for $|x| \le 1$ and are zero for $|x| > 1$. Thus, the regions of interest are found by plotting the four characteristics $x \pm t = \pm 1$ (Figure 1). The

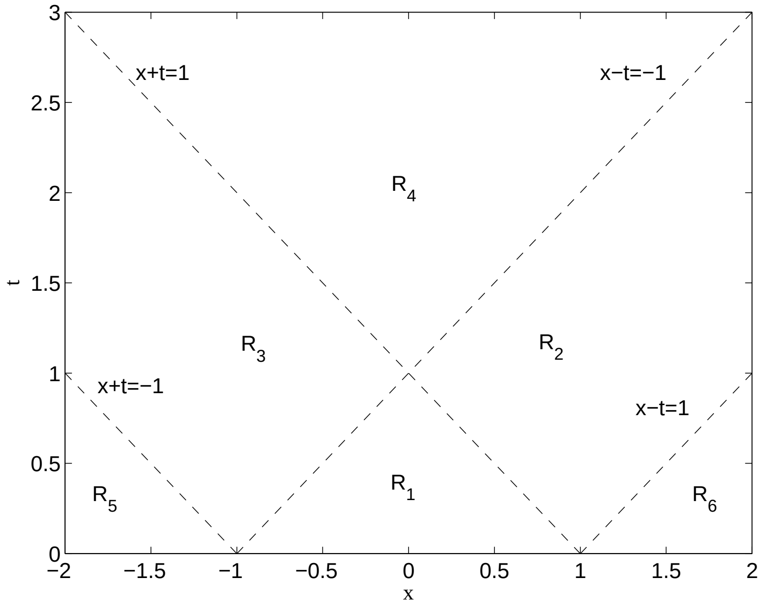

Figure 1: Regions of interest separated by four characteristics.

regions are identified in the plot, and are given mathematically by

$$\begin{aligned}
R_1 &= \{(x,t) : -1 \le x-t \le 1 \quad \text{and} \quad -1 \le x+t \le 1\} \\
R_2 &= \{(x,t) : -1 \le x-t \le 1 \quad \text{and} \quad x+t \ge 1\} \\
R_3 &= \{(x,t) : x-t \le -1 \quad \text{and} \quad -1 \le x+t \le 1\} \\
R_4 &= \{(x,t) : x-t \le -1 \quad \text{and} \quad x+t \ge 1\} \\
R_5 &= \{(x,t) : x+t \le -1\}, \\
R_6 &= \{(x,t) : x-t \ge 1\}
\end{aligned} \tag{33}$$

The regions determine where $x-t$ and $x+t$ are relative to ± 1, which tells us what part of the case functions $f(x)$ and $g(x)$ should be used.

Step 3. Consider the solution in each region. In R_1, combining the inequalities gives $|x \pm t| \le 1$ and hence from (32),

$$f(x-t) = F(x-t), \qquad f(x+t) = F(x+t), \qquad \int_{x-t}^{x+t} g(s)\,ds = \int_{x-t}^{x+t} G(s)\,ds$$

and

$$u(x,t) = \frac{1}{2}\left(F(x-t) + F(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} G(s)\,ds$$

In region R_2, we have $-1 \le x-t \le 1$ and $x+t \ge 1$ so that

$$f(x+t) = 0, \qquad f(x-t) = F(x-t), \qquad \int_{x-t}^{x+t} g(s)\,ds = \int_{x-t}^{1} G(s)\,ds.$$

and

$$u(x,t)=\frac{f(x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}g(s)\,ds=\frac{F(x-t)}{2}+\frac{1}{2}\int_{x-t}^{1}G(s)\,ds.$$

In region R_3, we have $-1\leq x+t\leq 1$ and $x-t\leq -1$ so that

$$f(x+t)=F(x+t),\qquad f(x-t)=0,\qquad \int_{x-t}^{x+t}g(s)\,ds=\int_{-1}^{x+t}G(s)\,ds.$$

and

$$u(x,t)=\frac{f(x+t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}g(s)\,ds=\frac{F(x+t)}{2}+\frac{1}{2}\int_{-1}^{x+t}G(s)\,ds.$$

In region R_4,

$$f(x+t)=0=f(x-t),\qquad \int_{x-t}^{x+t}g(s)\,ds=\int_{-1}^{1}G(s)\,ds$$

and

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t}g(s)\,ds=\frac{1}{2}\int_{-1}^{1}G(s)\,ds=const.$$

In regions R_5 and R_6, $f(x+t)=0=f(x-t)$ and $g(s)=0$ for $s\in[x-t,x+t]$, hence $u=0$. To summarize,

$$u(x,t)=\begin{cases}\frac{1}{2}\left(F(x-t)+F(x+t)\right)+\frac{1}{2}\int_{x-t}^{x+t}G(s)\,ds, & (x,t)\in R_1\\ \frac{F(x-t)}{2}+\frac{1}{2}\int_{x-t}^{1}G(s)\,ds, & (x,t)\in R_2\\ \frac{F(x+t)}{2}+\frac{1}{2}\int_{-1}^{x+t}G(s)\,ds, & (x,t)\in R_3\\ \frac{1}{2}\int_{-1}^{1}G(s)\,ds, & (x,t)\in R_4\\ 0 & (x,t)\in R_5,R_6\end{cases}\tag{34}$$

Step 4. For each specific time $t=t_0$, write the x-intervals corresponding to the sets R_n (i.e. the intersection of the set R_n with $\{t=t_0\}$, or in the figure above, where the line $t=t_0$ intersects the region R_n). As a check, we note that at $t=0$, the x intervals corresponding to the regions in (33) are

$$\begin{aligned}R_1\cap\{t=0\}&=\{-1\leq x\leq 1\}=\{|x|\leq 1\}\\ R_2\cap\{t=0\}&=\{x=1\}\\ R_3\cap\{t=0\}&=\{x=-1\}\\ R_4\cap\{t=0\}&=\varnothing\\ R_5\cap\{t=0\}&=\{x\leq -1\},\\ R_6\cap\{t=0\}&=\{x\geq 1\}\end{aligned}\tag{35}$$

and (34) becomes

$$u(x,0)=\begin{cases}F(x), & |x|\leq 1\\ 0, & |x|>1\end{cases}=f(x)$$

At $t = 1/2$, the x intervals corresponding to the regions in (33) are

$$\begin{aligned}
R_1 \cap \left\{t = \frac{1}{2}\right\} &= \left\{-\frac{1}{2} \leq x \leq \frac{3}{2} \quad \text{and} \quad -\frac{3}{2} \leq x \leq \frac{1}{2}\right\} = \left\{-\frac{1}{2} \leq x \leq \frac{1}{2}\right\} \\
R_2 \cap \left\{t = \frac{1}{2}\right\} &= \left\{-\frac{1}{2} \leq x \leq \frac{3}{2} \quad \text{and} \quad x \geq \frac{1}{2}\right\} = \left\{\frac{1}{2} \leq x \leq \frac{3}{2}\right\} \\
R_3 \cap \left\{t = \frac{1}{2}\right\} &= \left\{x \leq -\frac{1}{2} \quad \text{and} \quad -\frac{3}{2} \leq x \leq \frac{1}{2}\right\} = \left\{-\frac{3}{2} \leq x \leq -\frac{1}{2}\right\} \\
R_4 \cap \left\{t = \frac{1}{2}\right\} &= \left\{x \leq -\frac{1}{2} \quad \text{and} \quad x \geq \frac{1}{2}\right\} = \varnothing \\
R_5 \cap \left\{t = \frac{1}{2}\right\} &= \left\{x \leq -\frac{3}{2}\right\}, \\
R_6 \cap \left\{t = \frac{1}{2}\right\} &= \left\{x \geq \frac{3}{2}\right\}
\end{aligned} \tag{36}$$

At $t = 1$, the x intervals corresponding to the regions in (33) are

$$\begin{aligned}
R_1 \cap \{t = 1\} &= \{0 \leq x \leq 2 \quad \text{and} \quad -2 \leq x \leq 0\} = \{x = 0\} \\
R_2 \cap \{t = 1\} &= \{0 \leq x \leq 2 \quad \text{and} \quad x \geq 0\} = \{0 \leq x \leq 2\} \\
R_3 \cap \{t = 1\} &= \{x \leq 0 \quad \text{and} \quad -2 \leq x \leq 0\} = \{-2 \leq x \leq 0\} \\
R_4 \cap \{t = 1\} &= \{x \leq 0 \quad \text{and} \quad x \geq 0\} = \{x = 0\} \\
R_5 \cap \{t = 1\} &= \{x \leq -2\}, \\
R_6 \cap \{t = 1\} &= \{x \geq 2\}
\end{aligned} \tag{37}$$

At $t = 2$, the x intervals corresponding to the regions in (33) are

$$\begin{aligned}
R_1 &= \{1 \leq x \leq 3 \quad \text{and} \quad -3 \leq x \leq -1\} = \varnothing \\
R_2 &= \{1 \leq x \leq 3 \quad \text{and} \quad x \geq -1\} = \{1 \leq x \leq 3\} \\
R_3 &= \{x \leq 1 \quad \text{and} \quad -3 \leq x \leq -1\} = \{-3 \leq x \leq -1\} \\
R_4 &= \{x \leq 1 \quad \text{and} \quad x \geq -1\} = \{-1 \leq x \leq 1\} \\
R_5 &= \{x \leq -3\}, \\
R_6 &= \{x \geq 3\}
\end{aligned} \tag{38}$$

At this point, you usually want to draw the wave profiles for certain times $t = t_0$ (see problem set 3, question 3 for more).

5.6.3 Worked example

For an infinitely long string, consider giving the string zero initial displacement $u(x,0) = 0$ and initial velocity $u_t(x,0) = g(x)$. Suppose

$$g(x) = \begin{cases} \varepsilon \cos^2\left(\frac{\pi}{2}x\right), & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The ICs have the form considered above for $F(x) = 0$ and $G(x) = \varepsilon \cos^2\left(\frac{\pi}{2}x\right)$.

Step 1. D'Alembert's solution (31) becomes

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

Step 2. The regions are the same as those in (33) and plotted in the figure above.

Step 3. Determine $u(x,t)$ in each region. From (34), we have

$$u(x,t) = \begin{cases} \frac{1}{2}\int_{x-t}^{x+t} G(s)\,ds, & (x,t) \in R_1 \\ \frac{1}{2}\int_{x-t}^{1} G(s)\,ds, & (x,t) \in R_2 \\ \frac{1}{2}\int_{-1}^{x+t} G(s)\,ds, & (x,t) \in R_3 \\ \frac{1}{2}\int_{-1}^{1} G(s)\,ds, & (x,t) \in R_4 \\ 0 & (x,t) \in R_5, R_6 \end{cases}$$

Note that

$$\int_a^b \cos^2\left(\frac{\pi}{2}x\right) dx = \int_a^b \frac{1}{2}\left(1 + \cos(\pi x)\right) dx = \frac{b-a}{2} + \frac{\sin(\pi b) - \sin(\pi a)}{2\pi}$$

Thus

$$\begin{aligned} \frac{1}{\varepsilon}\int_{x-t}^{x+t} G(s)\,ds &= t + \frac{\sin(\pi(x+t)) - \sin(\pi(x-t))}{2\pi} = t + \frac{1}{\pi}\cos(\pi x)\sin(\pi t) \\ \frac{1}{\varepsilon}\int_{x-t}^{1} G(s)\,ds &= \frac{1-(x-t)}{2} - \frac{\sin(\pi(x-t))}{2\pi} \\ \frac{1}{\varepsilon}\int_{-1}^{x+t} G(s)\,ds &= \frac{1+x+t}{2} + \frac{\sin(\pi(x+t))}{2\pi} \\ \frac{1}{\varepsilon}\int_{-1}^{1} G(s)\,ds &= 1 \end{aligned}$$

Thus

$$\frac{u(x,t)}{\varepsilon} = \begin{cases} \frac{t}{2} + \frac{1}{2\pi}\cos(\pi t)\sin(\pi x), & (x,t) \in R_1 \\ \frac{1-(x-t)}{4} - \frac{\sin(\pi(x-t))}{4\pi}, & (x,t) \in R_2 \\ \frac{1+x+t}{4} + \frac{\sin(\pi(x+t))}{4\pi}, & (x,t) \in R_3 \\ \frac{1}{2}, & (x,t) \in R_4 \\ 0 & (x,t) \in R_5, R_6 \end{cases} \tag{39}$$

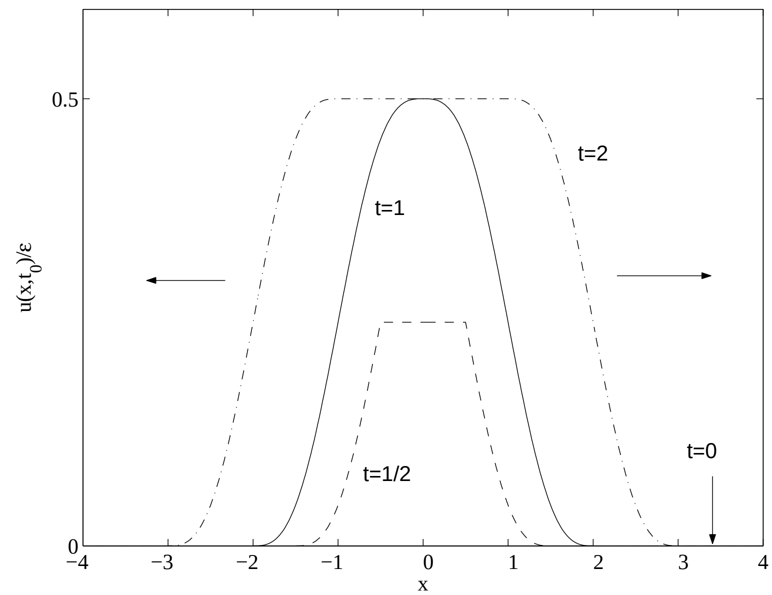

Figure 2: The profiles of the displacement $u(x,t)$ for the times $t = 0, 1/2, 1, 2$.

Step 4. We consider early, intermediate and later times, $t = 1/2$, 1, 2. At $t = 1/2$, the regions R_n are given by (36) and (39) becomes

$$\frac{u\left(x,\frac{1}{2}\right)}{\varepsilon} = \begin{cases} \frac{1}{4}, & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{4}\left(\frac{3}{2} - x\right) + \frac{\cos \pi x}{4\pi}, & \frac{1}{2} \le x \le \frac{3}{2} \\ \frac{1}{4}\left(\frac{3}{2} + x\right) + \frac{\cos \pi x}{4\pi}, & -\frac{3}{2} \le x \le -\frac{1}{2} \\ 0 & |x| \ge \frac{3}{2} \end{cases}$$

At $t = 1$, the regions R_n are given by (37) and (39) becomes

$$\frac{u\left(x,1\right)}{\varepsilon} = \begin{cases} \frac{2-x}{4} + \frac{\sin(\pi x)}{4\pi}, & 0 \le x \le 2 \\ \frac{2+x}{4} - \frac{\sin(\pi x)}{4\pi}, & -2 \le x \le 0 \\ 0 & |x| \ge 2 \end{cases}$$

At $t = 2$, the regions R_n are given by (38) and (39) becomes

$$\frac{u\left(x,2\right)}{\varepsilon} = \begin{cases} \frac{3-x}{4} - \frac{\sin(\pi x)}{4\pi}, & 1 \le x \le 3 \\ \frac{3+x}{4} + \frac{\sin(\pi x)}{4\pi}, & -3 \le x \le -1 \\ \frac{1}{2}, & -1 \le x \le 1 \\ 0 & |x| \ge 3 \end{cases}$$

In Figure 2, the profiles of the displacement $u(x,t)$ are plotted for the times $t = 0, 1/2, 1, 2$.

6 Waves on a finite string

[Oct 18, 2004]

Ref: §12.4, 12.5 Haberman

We now consider D'Alembert's solution for a finite string. The dimensionless problem with homogeneous Type I BCs (ends of string fixed) is

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(0,t) &= 0 = u(1,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \tag{40}$$

We found that this has solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)) \sin(n\pi x)$$

where

$$\begin{aligned} \alpha_n &= 2\int_0^1 f(x) \sin(n\pi x)\, dx \\ \beta_n &= \frac{2}{n\pi}\int_0^1 g(x) \sin(n\pi x)\, dx \end{aligned}$$

We wrote, equivalently, that

$$u(x,t) = P(x-t) + Q(x+t)$$

where

$$\begin{aligned} P(x-t) &= \frac{1}{2}\sum_{n=1}^{\infty} (\alpha_n \sin(n\pi(x-t)) + \beta_n \cos(n\pi(x-t))) \\ Q(x+t) &= \frac{1}{2}\sum_{n=1}^{\infty} (\alpha_n \sin(n\pi(x+t)) + \beta_n \cos(n\pi(x+t))) \end{aligned}$$

6.1 Zero initial velocity

The simplest ICs involve the case of zero initial velocity $u_t(x,0) = 0$ and a specified initial displacement $u(x,0) = f(x)$. In this case, $\beta_n = 0$ for all n and

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \cos(n\pi t) \sin(n\pi x) = P(x-t) + Q(x+t)$$

where

$$P(x-t)=\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\sin(n\pi(x-t)),\qquad Q(x+t)=\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\sin(n\pi(x+t)) \quad (41)$$

The odd 2-periodic extension of $f(x)$ is given by the sine series of $f(x)$,

$$\hat{f}(x)=\sum_{n=1}^{\infty}\alpha_n\sin(n\pi x),\qquad x\in\mathbb{R} \quad (42)$$

From (41) and (42), we can thus write $P(x-t)$ and $Q(x+t)$ as

$$P(x-t)=\frac{1}{2}\hat{f}(x-t),\qquad Q(x+t)=\frac{1}{2}\hat{f}(x+t)$$

so that

$$u(x,t)=\frac{1}{2}\left(\hat{f}(x-t)+\hat{f}(x+t)\right). \quad (43)$$

This looks like D'Alembert's solution for the infinite string - but $\hat{f}$ replaces $f(x)$ - so we'll see that some properties of the solution are different.

To see that (43) satisfies the BCs, note that since $\hat{f}$ is odd, $\hat{f}(-t)=-\hat{f}(t)$, and since $\hat{f}$ is 2-periodic and odd,

$$\hat{f}(1-t)=\hat{f}(-1-t)=-\hat{f}(1+t)$$

Thus,

$$u(0,t)=\frac{1}{2}\left(\hat{f}(-t)+\hat{f}(t)\right)=0$$
$$u(1,t)=\frac{1}{2}\left(\hat{f}(1-t)+\hat{f}(1+t)\right)=0$$

Therefore, (43) satisfies the Type I BCs (fixed ends).

To summarize, we have written the displacement of the string in terms of the odd 2-periodic extension of the initial condition $f(x)$. Eq. (43) is the solution for the finite string problem (40) with Type I BCs (fixed ends), zero initial velocity $u_t(x,0)=0$ and initial displacement $u(x,0)=f(x)$.

To draw the solution, note that x only goes from 0 to 1, but time $t\in[0,\infty)$. Thus $x+t\in[0,\infty)$ and $x-t\in[0,-\infty)$. We use the 2-periodic and odd properties of $\hat{f}$ to see what $u(x,t)$ looks like for $0<x<1$.

Example. Suppose $f(x)$ is a thin pulse symmetric about $x=1/2$ and with a maximum at $x=1/2$. The initial pulse breaks into forward and backward waves, each propagating with speed 1 in opposite directions. The peak of the forward wave (at $x=1/2$) first reaches the end of the string $x=1$ when

$$1-t=\frac{1}{2}\Longrightarrow t=\frac{1}{2}$$

At the same time, the peak of the backward wave reaches the end $x = 0$. Then what happens? On the first half of the string $0 \leq x \leq 1/2$ and times $1/2 \leq t \leq 1$,

$$-1 \leq x - t \leq 0$$

and thus

$$\hat{f}(x - t) = -\hat{f}(t - x) = -f(t - x)$$

Recall that $f(x)$ is the shape of the original pulse, so that on the first half of the string we have the negative of the forward wave. In other words, the forward wave went off the end $x = 1$ and reappeared at $x = 0$, UPSIDE DOWN (i.e. with negative sign)! On the second half of the string $1/2 \leq x \leq 1$ and times $1/2 \leq t \leq 1$,

$$1 \leq x + t \leq 2$$

and thus

$$\hat{f}(x + t) = -\hat{f}(x + t - 1) = -f(x + t - 1)$$

Thus the backward wave went off the end $x = 0$ at $t = 1/2$ and reappeared at the other end $x = 1$ (also at $t = 1/2$) and upside down (negative sign). At $t = 1$, the waves meet in the center,

$$u(x, 1) = \frac{1}{2}\left(\hat{f}(x - 1) + \hat{f}(x + 1)\right) = \frac{1}{2}\left(-\hat{f}(x) - \hat{f}(x)\right) = -\hat{f}(x) = -f(x)$$

and the displacement is the upside-down version of the initial displacement. The process continues in this manner. The characteristics $x \pm t$ criss-cross each other in the xt-plane. We can summarize the behavior using the following table:

time	x range	$x - t$ range	$x + t$ range	left half string	right half string
$0 \leq t \leq 1/2$	$[0, 1/2]$	$[0, 1]$		bkwd wave	
	$[1/2, 0]$		$[0, 1]$		fwd wave
$1/2 \leq t \leq 1$	$[0, 1/2]$	$[-1, 0]$		fwd wave, upside down	
	$[1/2, 1]$		$[1, 2]$		bkwd wave, upside down