Infinite Spatial Domain and the Fourier Transform

18.303 Linear Partial Differential Equations

Matthew J. Hancock

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Ref: Haberman, Ch 10

Consider the heat equation on an "infinite rod"

$$\begin{aligned} u_t &= \kappa u_{xx}, \qquad -\infty < x < \infty, \qquad t > 0 \\ u(x,0) &= f(x), \qquad -\infty < x < \infty. \end{aligned}$$

Since there is no boundary, we don't have so-called boundary conditions. However, we assume $f(x) \to 0$ as $|x| \to \infty$ and hence also assume that the temperature $u(x,t) \to 0$ as $|x| \to \infty$, for any time t. We proceed by separation of variables,

$$u(x,t) = X(x)\,T(t)$$

On substitution into the PDE, we obtain, as before,

$$\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda$$

and λ is constant since the l.h.s. depends only on t and the middle only on x. The individual problems for X and T are

$$\begin{aligned} X'' + \lambda X &= 0 \text{ (no BCs)} \\ T' &= -\kappa\lambda T \end{aligned}$$

Solving for T gives

$$T(t) = ce^{-\kappa\lambda t}$$

We have not said yet whether λ is negative, positive, or zero. Since we expect a heated rod to cool, we expect $|u(x,t)| < \infty$. If $\lambda < 0$, $T(t) \to \infty$ as $t \to \infty$ and

$X(x)$ blows up too, and hence λ must be non-negative. If $\lambda = 0$, then $T(t) = c$, $X = Ax + B$, and $A = 0$ to avoid a blowup. In fact, since we assumed $u \to 0$ as $|x| \to \infty$, we must have $A = 0$, so that $X = 0$. Thus $\lambda = 0$ leads to a trivial solution. It turns out that we can relax the conditions on $f(x)$ to allow $\lambda = 0$.

For $\lambda > 0$, we have

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

It is customary to let $\lambda = \omega^2$ (where $\omega > 0$), so that for each $\lambda = \omega^2 > 0$, we have a solution

$$u(x, t; \omega) = (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))\, e^{-\omega^2 \kappa t}$$

Since $\omega \in \mathbb{R}$, we must integrate, instead of summing, over all possible ω, to obtain the full solution:

$$u(x, t) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))\, e^{-\omega^2 \kappa t} d\omega \tag{1}$$

Imposing the initial condition $u(x, 0) = f(x)$ gives

$$f(x) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))\, d\omega \tag{2}$$

The concept of orthogonality for a discrete set of eigen-functions can be generalized provided $f(x)$ satisfies the following properties,

$$f(x) \text{ is piecewise smooth on every interval } [a, b] \text{ of the real line}$$

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

The second condition ensures $f(x)$ decreases fast enough for large $|x|$. If these properties of $f(x)$ hold, then $A(\omega)$ and $B(\omega)$ are given by

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\cos(\omega s)\, ds, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\sin(\omega s)\, ds.$$

Thus,

$$\begin{aligned}
u(x,t) &= \frac{1}{\pi}\int_0^\infty e^{-\omega^2\kappa t}\left(\cos(\omega x)\int_{-\infty}^\infty f(s)\cos(\omega s)\,ds \right. & (3)\\
&\qquad \left. + \sin(\omega x)\int_{-\infty}^\infty f(s)\sin(\omega s)\,ds\right)d\omega \\
&= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\omega^2\kappa t} f(s)\left(\cos(\omega x)\cos(\omega s)+\sin(\omega x)\sin(\omega s)\right)ds\,d\omega \\
&= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\omega^2\kappa t} f(s)\cos(\omega(x-s))\,ds\,d\omega \\
&= \int_{-\infty}^\infty\left(\frac{1}{\pi}\int_0^\infty e^{-\omega^2\kappa t}\cos(\omega(x-s))\,d\omega\right)f(s)\,ds \\
&= \int_{-\infty}^\infty K(s,x,t)\,f(s)\,ds & (4)
\end{aligned}$$

where

$$K(s,x,t) = \frac{1}{\pi}\int_0^\infty e^{-\omega^2\kappa t}\cos(\omega(x-s))\,d\omega \tag{5}$$

is called the Heat Kernel.

The integral for K can be calculated by defining $z = \omega\sqrt{\kappa t}$, so that

$$K(s,x,t) = \frac{1}{\pi\sqrt{\kappa t}}\int_0^\infty e^{-z^2}\cos(bz)\,dz, \qquad b = \frac{x-s}{\sqrt{\kappa t}}.$$

Let

$$I(b) = \int_0^\infty e^{-z^2}\cos(bz)\,dz \tag{6}$$

You can find $I(b)$ from Tables, e.g. Table of Integrals, Series & Products by Gradshteyn & Ryzhik. But here's how to calculate it: first, differentiate in b,

$$I'(b) = \int_0^\infty\left(-e^{-z^2}z\right)\sin(bz)\,dz$$

Integrating by parts gives

$$\begin{aligned}
I'(b) &= \left[\frac{e^{-z^2}}{b}\sin(bz)\right]_0^\infty - \frac{b}{2}\int_0^\infty e^{-z^2}\cos(bz)\,dz \\
&= 0 - \frac{b}{2}I(b)
\end{aligned}$$

Thus

$$\frac{dI}{I} = -\frac{b}{2}db$$

Integrating yields

$$I(b) = I(0)\,e^{-b^2/4}$$

and from (6),

$$I\left(0\right)=\int_{0}^{\infty}e^{-z^{2}}dz=\frac{\sqrt{\pi}}{2}.$$

Therefore,

$$I\left(b\right)=\frac{\sqrt{\pi}}{2}e^{-b^{2}/4}$$

Substituting $I\left(b\right)$ into (5) gives the Heat Kernel

$$K\left(s,x,t\right)=\frac{1}{\pi\sqrt{\kappa t}}I\left(b\right)=\frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{\left(x-s\right)^{2}}{4\kappa t}\right) \tag{7}$$

and temperature, from (4),

$$u\left(x,t\right)=\int_{-\infty}^{\infty}K\left(s,x,t\right)f\left(s\right)ds=\int_{-\infty}^{\infty}\frac{f\left(s\right)}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{\left(x-s\right)^{2}}{4\kappa t}\right)ds. \tag{8}$$

Since $K\left(s,x,t\right)$ decays exponentially as $\left|x\right|,\left|s\right|\to\infty$, then a sufficient condition for a solution is that $\left|f\left(s\right)\right|$ is bounded above by some constant as $\left|s\right|\to\infty$.

1 Hot spot on an infinite rod

As an example, consider the initial hot spot

$$u\left(x,0\right)=f\left(x\right)=\begin{cases} u_{0}/\varepsilon, & \left|x\right|\leq\varepsilon/2 \\ 0, & \left|x\right|>\varepsilon/2 \end{cases}$$

where $\varepsilon>0$. The temperature distribution is

$$\begin{aligned} u\left(x,t\right) &= \int_{-\infty}^{\infty}\frac{f\left(s\right)}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{\left(x-s\right)^{2}}{4\kappa t}\right)ds \\ &= \frac{u_{0}}{\varepsilon\sqrt{4\pi\kappa t}}\int_{-\varepsilon/2}^{\varepsilon/2}\exp\left(-\frac{\left(x-s\right)^{2}}{4\kappa t}\right)ds \end{aligned}$$

To evaluate this, we note that the error function is defined as

$$\operatorname{erf}\left(x\right)=\frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-x^{2}}dx$$

Let $z = (x - s)/\sqrt{4\kappa t}$. Then

$$\begin{aligned}
u(x,t) &= -\frac{u_0}{\varepsilon\sqrt{\pi}} \int_{(x+\varepsilon/2)/\sqrt{4\kappa t}}^{(x-\varepsilon/2)/\sqrt{4\kappa t}} \exp\left(-z^2\right) dz \\
&= -\frac{u_0}{\varepsilon\sqrt{\pi}} \left[\int_{(x+\varepsilon/2)/\sqrt{4\kappa t}}^{0} \exp\left(-z^2\right) dz + \int_0^{(x-\varepsilon/2)/\sqrt{4\kappa t}} \exp\left(-z^2\right) dz\right] \\
&= \frac{u_0}{\varepsilon\sqrt{\pi}} \left[\int_0^{(x+\varepsilon/2)/\sqrt{4\kappa t}} \exp\left(-z^2\right) dz - \int_0^{(x-\varepsilon/2)/\sqrt{4\kappa t}} \exp\left(-z^2\right) dz\right] \\
&= \frac{u_0}{2\varepsilon} \left[\operatorname{erf}\left(\frac{x+\varepsilon/2}{\sqrt{4\kappa t}}\right) - \operatorname{erf}\left(\frac{x-\varepsilon/2}{\sqrt{4\kappa t}}\right)\right]
\end{aligned}$$

Plots are given below in Figures 1 to 4. The solution $u(x,t)$ is just the sum of shifted $\operatorname{erf} x$ functions, and hence is straightforward to plot once you know what $\operatorname{erf} x$ looks like (see Figures 1, 2). The error function is $2/\sqrt{\pi}$ times the area under e^{-s^2} from $s = 0$ to $s = x$ (Figure 1). Thus as $s \to \infty$, the area is $\sqrt{\pi}/2$ since $\operatorname{erf}(\infty) = 1$. To obtain Figure 3, note that

$$u_t = -\frac{u_0}{4\varepsilon\sqrt{\pi\kappa}t^{3/2}} \left[\left(x + \frac{\varepsilon}{2}\right) \exp\left(-\frac{(x+\varepsilon/2)^2}{4\kappa t}\right) - \left(x - \frac{\varepsilon}{2}\right) \exp\left(-\frac{(x-\varepsilon/2)^2}{4\kappa t}\right)\right]$$

and

$$u_t < 0, \qquad |x| < \varepsilon/2$$

For $|x| > \varepsilon/2$, u increases and then decreases. Thus $u_t = 0$ when

$$\ln\frac{x+\varepsilon/2}{x-\varepsilon/2} = \frac{\varepsilon x}{2\kappa t}$$

and hence when

$$t = t_* = \frac{\varepsilon x}{2\kappa \ln\frac{x+\varepsilon/2}{x-\varepsilon/2}}.$$

2 Fourier Transform

We now introduce the Fourier Transform and show how it is related to the solution of the Heat Problem on an infinite domain. Consider a function $g(x)$ defined on $-\infty < x < \infty$ that satisfies the properties listed above, namely,

$$g(x) \text{ is piecewise smooth on every interval } [a,b] \text{ of the real line} \tag{9}$$

$$\int_{-\infty}^{\infty} |g(x)|\, dx < \infty. \tag{10}$$

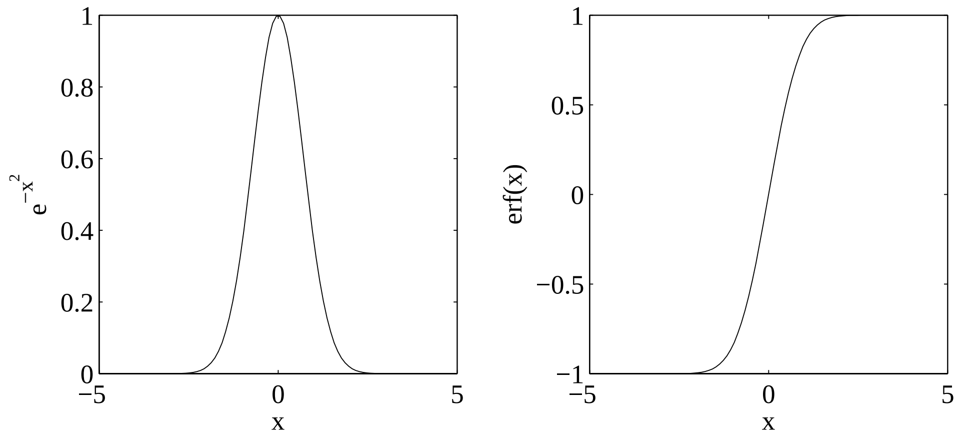

Figure 1: Plots of e^{-x^2} and $\mathrm{erf}(x)$.

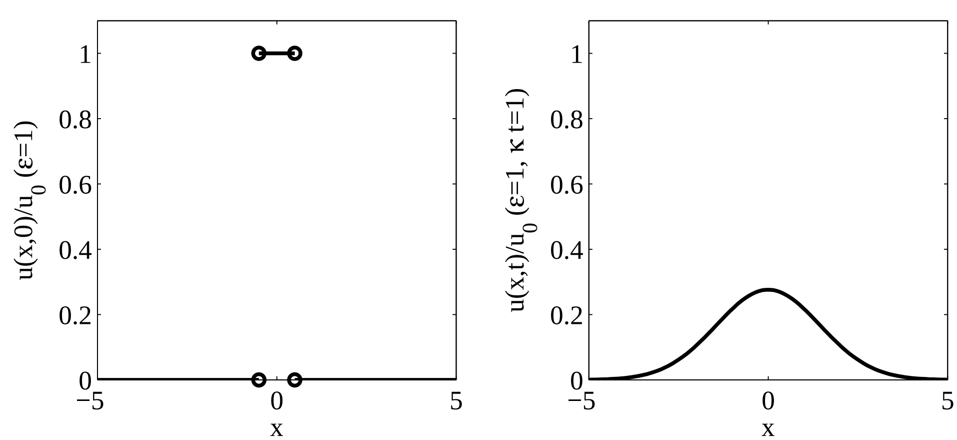

Figure 2: Plot of $u(x,t)$ for $t=0$ and $\kappa t=1$, with $\varepsilon=1$.

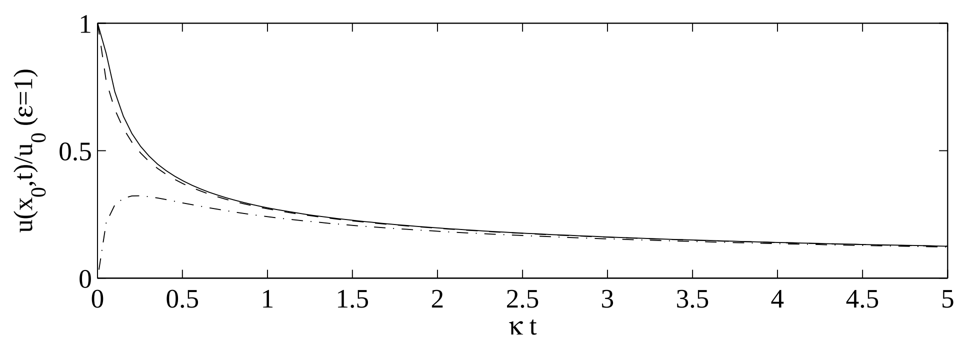

Figure 3: Plot of $u(x,t)$ for $x=0$ (solid), $x=\pm 1/4$ (dash) and $x=\pm 3/4$ (dot-dash), for $\varepsilon=1$.

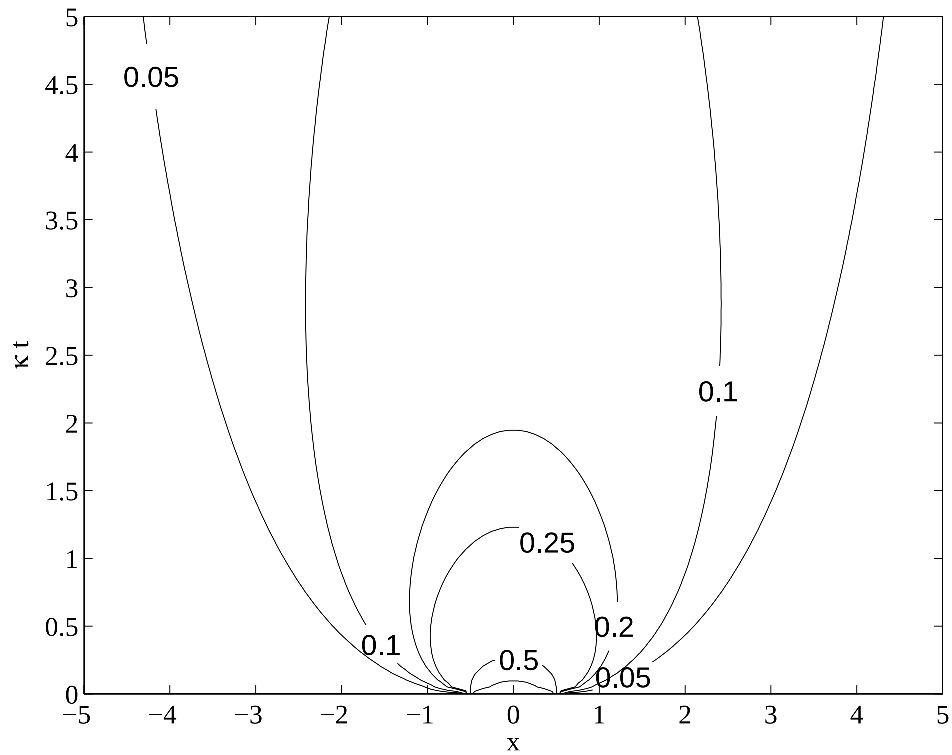

Figure 4: Plot of the level curves of $u(x,t)$, i.e. the curves on which $u(x,t) = const$. Numbers indicate the value of $u(x,t)/u_0$ on the level curve. Here $\varepsilon = 1$.

The Fourier Transform $G(\omega)$ of the function $g(x)$ is defined as

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\, e^{i\omega x} dx \tag{11}$$

The Inverse Fourier Transform $g(x)$ of a function $G(\omega)$ is given by

$$g(x) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega x} d\omega \tag{12}$$

2.1 Fourier transform and the solution to the heat equation

To relate the solution of the Heat Problem on an infinite domain $-\infty < x < \infty$ to the Fourier Transform, we must make some manipulations to our solution. In particular, Eq. (1) becomes

$$\begin{aligned} u(x,t) &= \int_0^{\infty} \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right) e^{-\omega^2\kappa t} d\omega \\ &= \int_0^{\infty} F_1(\omega)\, e^{i\omega x} e^{-\omega^2\kappa t} d\omega + \int_0^{\infty} F_2(\omega)\, e^{-i\omega x} e^{-\omega^2\kappa t} d\omega, \end{aligned} \tag{13}$$

where

$$F_1(\omega) = \frac{1}{2}\left(A(\omega) - iB(\omega)\right), \qquad F_2(\omega) = \frac{1}{2}\left(A(\omega) + iB(\omega)\right).$$

We now allow ω to take all real values, i.e., $-\infty < \omega < \infty$, and continue to manipulate the solution $u(x,t)$. Making the change of variable $\omega \to -\omega$ in the first integral of (13) gives

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} F_1(-\omega)\, e^{-i\omega x} e^{-\omega^2\kappa t} d\omega + \int_0^{\infty} F_2(\omega)\, e^{-i\omega x} e^{-\omega^2\kappa t} d\omega \\ &= \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x} e^{-\omega^2\kappa t} d\omega, \end{aligned} \tag{14}$$

where

$$F(\omega) = \begin{cases} F_2(\omega) & \omega \geq 0 \\ F_1(-\omega) & \omega < 0 \end{cases}$$

Furthermore, at $t = 0$, we have

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x} d\omega. \tag{15}$$

Comparing Eq. (15) to (12) shows that $f(x)$ is the Inverse Fourier Transform of $F(\omega)$, or equivalently, $F(\omega)$ is the Fourier Transform of the initial temperature distribution $f(x)$. From (11), we have

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx. \tag{16}$$

Therefore, given an initial temperature distribution $f(x)$ that satisfies conditions (9) and (10), we find the Fourier Transform $F(\omega)$ of $f(x)$, and the temperature distribution at each point in time is then given by (14). The integral in (16) can generally be found in tables (e.g. Table 10.4.1 in Haberman, p. 468).

2.2 Fourier transform of the heat equation

We now define the Fourier Transform (FT) of a function $u(x,t)$ as an operator:

$$\mathcal{F}[u] = \overline{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x} dx \tag{17}$$

Thus, the FT $\mathcal{F}$ maps a function of (x,t) to a function of (ω,t). To transform the Heat Equation, we must consider how the FT maps derivatives. Note that

$$\begin{aligned}\mathcal{F}\left[\frac{\partial u}{\partial t}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}(x,t)\, e^{i\omega x}dx \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}dx\right) = \frac{\partial}{\partial t}\mathcal{F}[u] = \frac{\partial}{\partial t}\overline{U}(\omega,t)\end{aligned}$$

Also, integration by parts and the fact that $u(x,t) \to 0$ as $|x| \to \infty$ allow us to calculate the FT of $\partial u/\partial x$,

$$\begin{aligned}\mathcal{F}\left[\frac{\partial u}{\partial x}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}(x,t)\, e^{i\omega x}dx \\ &= \frac{1}{2\pi}\left[e^{i\omega x}u(x,t)\right]_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\left(i\omega e^{i\omega x}\right)dx \\ &= -i\omega\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}dx\right) \\ &= -i\omega\mathcal{F}[u] = -i\omega\overline{U}(\omega,t).\end{aligned}$$

Thus, the FT of an x-derivative of a function is mapped to $-i\omega$ times the FT of the function. Hence

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -i\omega\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = (-i\omega)^2\,\mathcal{F}[u] = -\omega^2\overline{U}(\omega,t)$$

Thus the FT of the Heat Equation $u_t = \kappa u_{xx}$ is

$$\frac{\partial}{\partial t}\overline{U}(\omega,t) = -\kappa\omega^2\overline{U}(\omega,t).$$

Hence, the Fourier Transform maps the heat equation, a PDE, to a first order ODE! Integrating in time gives

$$\overline{U}(\omega,t) = C(\omega)\, e^{-\kappa\omega^2 t} \tag{18}$$

where $C(\omega)$ is an arbitrary function, due to partial integration with respect to time. Setting $t = 0$ in (17) and (18) gives

$$C(\omega) = \overline{U}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, 0) e^{i\omega x} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx.$$

We could substitute $C(\omega)$ back into (18) and use the Inverse Fourier Transform to obtain $u(x,t)$. The solution would be the same as that for separation of variables (recall that uniqueness holds). However, $u(x,t)$ can be obtained almost immediately using a result called the Convolution Theorem.

One important FT is that of the Gaussian distribution,

$$\mathcal{F}\left[e^{-\beta x^2}\right] = \frac{1}{\sqrt{4\pi\beta}} e^{-\omega^2/4\beta}$$

Thus, the FT of a Gaussian is a Gaussian. In particular, the IFT of a Gaussian is

$$\mathcal{F}^{-1}\left[e^{-\alpha\omega^2}\right] = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha} \tag{19}$$

2.3 Convolution Theorem

Ref: Haberman p. 466 (§10.4.3)

Convolution Theorem Suppose that $F(\omega)$ and $G(\omega)$ are the Fourier Transforms of $f(x)$ and $g(x)$, then

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx, \qquad G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x) e^{i\omega x} dx, \\ f(x) &= \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega, \qquad g(x) = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} d\omega. \end{aligned}$$

Let $H(\omega) = F(\omega) G(\omega)$. The Inverse Fourier Transform (IFT) of $H(\omega)$ is

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) g(x-s) ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s) f(x-s) ds \tag{20}$$

The first integral in (20) is called the convolution of $f(x)$ and $g(x)$. In other words, the IFT of the product of two FTs is the $1/2\pi$ times the convolution of the two functions.

Proof: The IFT of $H(\omega)$ is

$$h(x) = \int_{-\infty}^{\infty} H(\omega) e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega x} d\omega$$

Substituting the IFT of $F(\omega)$ gives

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) e^{i\omega s} ds\right) G(\omega) e^{-i\omega x} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s) G(\omega) e^{-i\omega x} e^{i\omega s} ds d\omega \end{aligned}$$

Assuming we can interchange the order of integration (we can provided $\int_{-\infty}^{\infty} |f(s)|\, ds$ and $\int_{-\infty}^{\infty} |G(\omega)|\, d\omega$ are finite and $f(s)$, $G(\omega)$ are smooth) we have

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left(\int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega(x-s)} d\omega \right) ds$$

Note that

$$g(x-s) = \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega(x-s)} d\omega$$

and hence

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds.$$

This is the first integral in (20). To obtain the second, we make the transformation $w = x - s$, so the integral becomes

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(w)\, f(x-w)\, dw.$$

Since w is a dummy variable, we can replace it with s to obtain the second integral in (20). ■

To apply the convolution theorem to the Heat Equation, we note that the FT of the solution $u(x,t)$ is

$$\overline{U}(\omega, t) = C(\omega)\, e^{-\kappa\omega^2 t}$$

Since the IFT of $C(\omega)$ is $f(x)$ and the IFT of $e^{-\kappa\omega^2 t}$ is, by (19), $\sqrt{\pi/(\kappa t)}e^{-x^2/(4\kappa t)}$, then, by the Convolution Theorem,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{\pi}{\kappa t}} e^{-(x-s)^2/(4\kappa t)} ds$$

which agrees with the solution we found using separation of variables, Eq. (8).

2.4 Heat equation on a semi-infinite domain

Consider the heat equation on a semi-infinite domain $0 \le x < \infty$,

$$\begin{aligned} u_t &= \kappa u_{xx}, \qquad 0 \le x < \infty \\ u(x,0) &= f(x), \qquad 0 \le x < \infty \end{aligned}$$

and $f(x)$, $u(x,t)$ approach zero fast enough as $|x| \to \infty$ so that the integral

$$\int_{-\infty}^{\infty} |u(x,t)|\, dx$$

is finite. Also, a boundary condition is imposed at $x = 0$. Either a Type I BC (fixed temperature),

$$u(0,t) = 0, \qquad t > 0, \tag{21}$$

or Type II BC (insulated),

$$u_x(0,t) = 0, \qquad t > 0. \tag{22}$$

To solve this problem, we recall the temperature distribution on the infinite domain $-\infty < x < \infty$ due to an initial temperature $\widetilde{f}(x)$ was given by

$$u(x,t) = \int_{-\infty}^{\infty} K(s,x,t)\, \widetilde{f}(s)\, ds$$

where the Heat Kernel is defined as

$$K(s,x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-s)^2}{4\kappa t}\right).$$

Note that at $x = 0$, the Heat Kernel is even in s, i.e. $K(s,0,t) = K(-s,0,t)$. Also,

$$u(0,t) = \int_{-\infty}^{\infty} K(s,0,t)\, \widetilde{f}(s)\, ds$$

Thus $u(0,t) = 0$ if $\widetilde{f}(s)$ is odd. Therefore, the solution to the Heat Problem on the semi-infinite domain $0 \le x < \infty$ with zero temperature at $x = 0$ ($u(0,t) = 0$) is

$$u(x,t) = \int_{-\infty}^{\infty} K(s,x,t)\, \widetilde{f}(s)\, ds$$

where $\widetilde{f}(s)$ is the odd extension of $f(s)$, i.e.

$$\widetilde{f}(s) = \begin{cases} f(s) & s > 0 \\ 0 & s = 0 \\ -f(-s) & s < 0 \end{cases}$$

Similarly, note that the x-derivative of the Heat Kernel is odd at $s = 0$,

$$K_x(s,0,t) = \frac{s}{2\kappa t\sqrt{4\pi\kappa t}} \exp\left(-\frac{s^2}{4\kappa t}\right) = -K_x(-s,0,t),$$

and hence the solution to the Heat Problem on the semi-infinite domain $0 \le x < \infty$ with an insulated BC at $x = 0$ ($u_x(0,t) = 0$) is

$$u(x,t) = \int_{-\infty}^{\infty} K(s,x,t)\, \widetilde{f}(s)\, ds$$

where $\widetilde{f}(s)$ is the even extension of $f(s)$, i.e.

$$\widetilde{f}(s) = \begin{cases} f(s) & s > 0 \\ 0 & s = 0 \\ f(-s) & s < 0 \end{cases}$$

3 Fourier Transform solution to Laplace's Equation

Suppose the temperature of an infinite wall is kept at $f(x)$, for $-\infty < x < \infty$. Find the steady-state temperature in the region adjoining the wall, $y > 0$. The steady-state temperature satisfies Laplace's equation,

$$\nabla^2 u_E = 0, \qquad -\infty < x < \infty, \qquad y > 0.$$

The BCs are

$$u_E = f(x), \qquad -\infty < x < \infty, \qquad y = 0,$$

$$\lim_{y\to\infty} u_E(x,y) = 0, \qquad \lim_{|x|\to\infty} u_E(x,y) = 0.$$

We employ the Fourier transform in x,

$$\mathcal{F}[g(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x,y)\, e^{i\omega x} dx$$

We define $U_E(\omega, y) = \mathcal{F}[u_E(x,y)]$. As before, we have

$$\mathcal{F}[u_{Exx}] = -\omega^2 \mathcal{F}[u_E] = -\omega^2 U_E(\omega,y), \qquad \mathcal{F}[u_{Eyy}] = \frac{\partial^2}{\partial y^2}\mathcal{F}[u_E] = \frac{\partial^2}{\partial y^2} U_E(\omega,y).$$

Hence Laplace's equation for the steady-state temp $u_E(x,y)$ becomes

$$\frac{\partial^2}{\partial y^2} U_E(\omega,y) - \omega^2 U_E(\omega,y) = 0$$

Solving the ODE and being careful about the fact that ω can be positive or negative, we have

$$U_E(\omega,y) = c_1(\omega)\, e^{-|\omega| y} + c_2(\omega)\, e^{|\omega| y}$$

where $c_1(\omega)$, $c_2(\omega)$ are arbitrary functions. Since the temperature must vanish as $y \to \infty$, we must have $c_2(\omega) = 0$. Thus

$$U_E(\omega,y) = c_1(\omega)\, e^{-|\omega| y} \tag{23}$$

Imposing the BC at $y = 0$ gives

$$c_1(\omega) = U_E(\omega,0) = \mathcal{F}[u_E(x,0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_E(x,0)\, e^{i\omega x} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx.$$

Note that the IFT of $e^{-|\omega| y}$ is

$$\begin{aligned}
\mathcal{F}^{-1}\left[e^{-|\omega| y}\right] &= \int_{-\infty}^{\infty} e^{-|\omega| y} e^{-i\omega x} d\omega = \int_{-\infty}^{0} e^{\omega(y-ix)} d\omega + \int_{0}^{\infty} e^{-\omega(y+ix)} d\omega \\
&= \left[\frac{e^{\omega(y-ix)}}{y-ix}\right]_{-\infty}^{0} + \left[\frac{e^{-\omega(y+ix)}}{-(y+ix)}\right]_{0}^{\infty} \\
&= \frac{1}{y-ix} + \frac{1}{y+ix} = \frac{2y}{x^2+y^2}
\end{aligned}$$

Therefore, applying the Convolution Theorem to (23) with $\mathcal{F}^{-1}\left[c_1(\omega)\right] = f(x)$ and $\mathcal{F}^{-1}\left[e^{-|\omega|y}\right] = 2y/\left(x^2+y^2\right)$ gives

$$u_E(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\frac{2y}{(x-s)^2+y^2}\,ds.$$